# Optimal bounds for local volumes of 3fold singularities

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/$\mathbb{C}$

## § Intro

$x \in X$ klt singularity.

log discrepancy, volume.

min

C. Li $\quad \widehat{\mathrm{vol}}(x,X) = \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v) = \inf_{v} A_X(v)^n \cdot \mathrm{vol}(v).$

local volume

normalized volume of valuation.

$\mathrm{Val}_{X,x}$ = space of real valuations of $K(X)$ centered at $x$.

This leads to a local K-stability theory for klt singularities.

## Basic properties.

- $\widehat{\mathrm{vol}}(0, \mathbb{A}^n) = n^n$. computed by $\mathrm{ord}_0$.

  (de Fernex-Ein-Mustaţă, C. Li).

- $0 < \widehat{\mathrm{vol}}(x, X) \leq n^n$, "=" iff $x \in X$ is smooth.
  [L.-Xu]

- $\widehat{\mathrm{vol}}(x, X)$ is lower semi-continuous in families.
  [L.-Blum]. and constructible [Xu].

- $\widehat{\mathrm{vol}}(x, X)$ is an alg. # [Xu-Zhuang]
  but can be irrational (cone over $\mathbb{F}_1$)

- $(y \in Y) \xrightarrow{\pi} (x \in X)$ (Galois) quasi-~~étale~~, $\widehat{\mathrm{vol}}(y, Y) = \deg(\pi) \cdot \widehat{\mathrm{vol}}(x, X)$.

**Thm** (Xu–Zhuang). Fix $\dim X = n$, Fix $\forall c > 0$.

Then $\{ x \in X \mid \widehat{\mathrm{vol}}(x, X) \geq c \}$ satisfies some bounded properties.

Precisely, $(x \in X) \overset{v_{\min}}{\rightsquigarrow} (x_0 \in X_0)$ K-ss Fano cone

bounded.

**Cor**. The set $\{ \widehat{\mathrm{vol}}(x, X) \mid \dim X = n \}$ is discrete away from $0$.

Conj. (ODP gap)

If $x \in X$ singular, then $\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n$
$$= \widehat{\mathrm{vol}}(\mathrm{ODP}).$$

"=" iff $x \in X$ is ODP.

True in dim $\leq 3$. (L.-Xu for $n = 3$)

Open in dim $\geq 4$.

Conj $\Rightarrow$ K-moduli space of cubic n-folds $\cong$ GIT.

$\Rightarrow$ all smooth cubic n folds have KE metric.

**Thm A** (L.'25) $x \in X$ 3 fold klt singularity

Suppose $\widehat{\mathrm{vol}}(x, X) \geq 9$.

$\Uparrow$ Then either $x \in X$ is a hypersurface sing of (or smooth) $cA_1$ or $cA_2$

or $x \in X$ is a cyclic quotient sing. of order $\leq 3$.

**Thm B** (L.'25) $x \in X$ Gor. can. 3fold sing not hypersurface.

Then $\widehat{\mathrm{vol}}(x, X) \leq 9$, "=" iff $\frac{1}{3}(1,1,1)$.

Applications. K-moduli of Fano 3folds.

$X$ in some family of K-moduli stack.

Mori-Mukai

$V = (-K_X)^3$.

Then

(1) $V \geq 26 \Rightarrow x \in X$ is of type $cA_1$.

(2) $V \geq 22 \Rightarrow x \in X$ is of type $cA_1$, isolated $cA_2$, or $D_\infty$.

(3) $V \geq 11 \Rightarrow$ non-Gor $x \in X$ is a cyclic quotient of $cA_{\leq 2}$-sing.

<u>Thm</u>. For any 3fold klt sing $x \in X$,

$$\widehat{\mathrm{vol}}(x, X) \leq 9 \cdot \mathrm{mld}(x, X).$$

"=" iff $x \in X$ is cyclic quotient $\frac{1}{r}(1,1,1)$.

(cone over $\mathbb{P}^2$ pol. $\mathcal{O}(r)$).

<u>Thm</u> B (L.'25) $x \in X$ Gor. can. 3fold sing not hypersurface.

Then $\widehat{\mathrm{vol}}(x, X) \leq 9$, "=" iff $\frac{1}{3}(1,1,1)$.

<u>Thm</u> (L-Xu'19) $x \in X$ klt 3fold sing not smooth

Then $\widehat{\mathrm{vol}}(x, X) \leq 16$, "=" iff $A_1$.

<u>Ex</u>. compound du Val

$cA_1$: $x^2+y^2+z^2+w^{k+1}=0$ $\quad \widehat{\mathrm{vol}} \in \{16, \frac{125}{9}, \frac{27}{2}\}$
($A_k$-sing) $\quad k=1, 2, \geq 3$.

$cA_2$: $x^2+y^2+f(z,w)=0$ $\quad \{\widehat{\mathrm{vol}}\} \supseteq \{\frac{32}{3}, 6\sqrt{3}, \cdots, 9\}$.
$\mathrm{mult}_0 f = 3$.

**Lem.** $y \in Y \xrightarrow[\text{not } \cong]{\text{crep bir}} x \in X$

Then $\widehat{\mathrm{vol}}(y, Y) > \widehat{\mathrm{vol}}(x, X)$.

**Def.** $x \in X$ $\overset{\text{Gor.}}{\text{canonical}}$ sing. 3 fold. (klt)

$\mathrm{crex}(x, X)$ = # of exc. divisors $E$ over $X$

centered at $x$ s.t. $A_X(E) = 1$.

crex is always finite

**Thm.** $x \in X$ satisfies $\overset{\text{(terminal)}}{\mathrm{crex}(x, X) = 0}$ iff $x \in X$ is $\overset{\text{(isolated cDV)}}{\text{cDV}}$.

For cDV sing (more generally hypersurfaces)

$\widehat{\mathrm{vol}}$ can be bounded by monomial valuations

(weighted blow ups).

__Next__. need to bound $\widehat{\mathrm{vol}}$ if $\mathrm{crex} > 0$.

__Rough idea__: $x \in X$ Gor. can. 3fold, $\mathrm{crex}(x, X) > 0$.

Then $\exists\ Y \xrightarrow[\text{bir}]{\text{crep.}} X$ extract all crepant exc. div.

Then $Y$ has only cDV sing. ($Y$ could be smooth)

Let $E_1 \subseteq \mathrm{Exc}(Y/X)$ be a prime divisor

(crepant exc over $x \in X$).

Run $E_1$-MMP$/X$ to contract $E_1$

$$Y \overset{\text{flips}}{\dashrightarrow} Y_1 \xrightarrow{\text{contract } E_1} Y'.$$

2 cases: (1) $E_1$ gets contracted to a pt $y' \in Y'$.

$\mathrm{crex}(y', Y') = 1$. $\widehat{\mathrm{vol}}(y', Y') \geq \widehat{\mathrm{vol}}(x, X)$.

(2) $E_1$ gets contracted to a curve $C \ni y'$

$Y'$ has $A_1 \times \mathbb{C}$-sing along $C$. $\widehat{\mathrm{vol}}(y', Y') = \frac{27}{2}$.

Next, assume $\operatorname{crex}(x, X) \geq 2$.

Need to run $E_2$-MMP on $Y'$ to get

$$\begin{array}{ccccccc} Y_1 & \to & Y' & \overset{\text{flips}}{\dashrightarrow} & Y_2 & \longrightarrow & Y'', \\ \cup\!\!\!\mid & & \subseteq\ \cup\!\!\!\mid & & & & \\ E_1 & \to & C \subseteq E_2 & & & & \end{array}$$

$\Rightarrow$ reduce to $\operatorname{crex} = 1$ or $2$.

Need $C \subseteq E_2$ which is OK if $X$ is $\mathbb{Q}$-factorial.

(Need to contract a single curve in a $\mathbb{Q}$-factorial mod.)
need Artin approximation.

$c r^{ex} = 1$, $\quad Y \longrightarrow X$ extract $E$.

s.t. $-E$ ample $/ X$.

$Y$ is cDV, $E$ is Cartier in $Y$.

By adjunction, $E$ is an integral Gor. del Pezzo surface.

[Reid '94]: classify $E$. $\Rightarrow \widehat{\mathrm{vol}}(x, X) \leq (-K_E)^2$.

2 families: $\mathbb{F}_m$ glue neg. sec. with a fiber.

$\mathbb{F}_m$ glue neg. sec to itself via involution.